

THE ASYMPTOTIC EXPANSION OF THE HEAT KERNEL ON A COMPACT LIE GROUP

SEUNGHUN HONG

ABSTRACT. Let G be a compact connected Lie group equipped with a bi-invariant metric. We calculate the asymptotic expansion of the heat kernel of the laplacian on G and the heat trace using Lie algebra methods. The Duflo isomorphism plays a key role.

CONTENTS

1. Introduction	1
2. Preliminaries	2
3. The Asymptotic Expansion of the Heat Kernel of the Laplacian on a Compact Lie Group	5
References	7

1. INTRODUCTION

(1.1) The laplacian Δ on a compact riemannian manifold M depends only on the metric on M . Conversely, one can deduce the metric from the laplacian. Hence, it is reasonable to expect the spectrum $\text{Sp}(\Delta)$ of the laplacian to be constrained by the geometry and vice versa. The first result in this vein is Weyl's law [11, 12] which states that the number $N(\lambda)$ of the eigenvalues of the laplacian less than λ satisfies the asymptotic equality

$$\frac{N(\lambda)}{\lambda^{n/2}} = \frac{\text{vol}(\Omega)}{(4\pi)^{n/2}\Gamma(\frac{n}{2} + 1)} + O(1/\lambda)$$

as $\lambda \rightarrow \infty$, where Ω is a bounded open subset of \mathbb{R}^2 or \mathbb{R}^3 on which the laplacian is defined. Gårding [5] proved the higher-dimensional case (for generic elliptic operators). On closed riemannian manifolds, the same law was proved for the laplacian by Duistermaat and Guillemin [4], and for generic elliptic operators by Minakshisundaram and Pleijel [9].

Weyl's law can be reformulated as an asymptotic behavior of the function

$$Z(t) = \sum_{k=1}^{\infty} e^{-t\lambda_k}$$

where $-\lambda_k$ denotes the eigenvalue of the k th eigenfunction of the laplacian. This function resembles the “partition function” in physics—a function that is often invariant under the symmetry of the physical system it describes. It is the trace of the heat diffusion operator

$$e^{t\Delta} = \begin{pmatrix} e^{-t\lambda_1} & & & \\ & e^{-t\lambda_2} & & \\ & & e^{-t\lambda_3} & \\ & & & \ddots \end{pmatrix}.$$

2010 *Mathematics Subject Classification*. Primary 58J05, 58J35, 58J37, 58J50, 58J60; Secondary 35K08.

Key words and phrases. compact Lie groups, Laplace-Beltrami operator, heat kernel expansion, heat trace expansion, Duflo isomorphism.

The author wishes to express his heartfelt gratitude to his advisor, Professor N. Higson, for the kind guidance and advice.

The relation between $Z(t)$ and the number of eigenfunctions becomes evident as we consider the limit $t \rightarrow 0+$; in that limit, the partial sum

$$\sum_{k=1}^K e^{-t\lambda_k}$$

converges to K , the number of the eigenvalues from λ_1 to λ_K . Weyl's law can be shown to be equivalent to the asymptotic law

$$(1.2) \quad t^{n/2} Z(t) \sim \frac{\text{vol}(\Omega)}{(4\pi)^{n/2}} + O(t)$$

as $t \rightarrow 0+$. Minakshisundaram and Pleijel [9] showed that the partition function has an asymptotic expansion

$$Z(t) \sim \left(\frac{1}{4\pi t} \right)^{\dim M/2} (a_0 + a_1 t + a_2 t^2 + \dots)$$

as $t \rightarrow 0+$. McKean and Singer [8] calculated a_0, a_1, a_2 and, in particular, showed that a_0 is the riemannian volume of M and a_1 is $\frac{1}{6} \int_M S$ where S is the scalar curvature of M . The higher order coefficients are extremely hard to calculate in general.

Our goal is to consider the asymptotic expansion of $Z(t)$ in the case where M is a compact connected Lie group G , equipped with a metric that is invariant under the left and right-translations. We follow the heat kernel method, that is, we calculate the asymptotic expansion of the heat kernel $k_t(x, y)$ of the laplacian. The asymptotic expansion for $Z(t)$ can then be obtained from the relation

$$Z(t) = \int_G k_t(x, x) \text{vol}(x).$$

Here $\text{vol}(x)$ is the riemannian volume form. Our motivation comes from the expectation that the high degree of symmetry will substantially simplify the calculations.

Our strategy is to utilize the tight connection between G and its Lie algebra \mathfrak{g} . The key ingredient is the Duflo isomorphism

$$\text{Duf} : S(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathcal{Z}(\mathfrak{g}).$$

The space $S(\mathfrak{g})^{\mathfrak{g}}$ can be identified with the constant coefficient differential operators on \mathfrak{g} , and $\mathcal{Z}(\mathfrak{g})$ can be identified with the bi-invariant differential operators on G . Owing to the bi-invariance of the metric on G , the laplacian Δ_G on G is a bi-invariant differential operator; and it corresponds to the Casimir element in $\mathcal{Z}(\mathfrak{g})$. The operator on \mathfrak{g} corresponding to the preimage of the Casimir under the Duflo isomorphism is (not surprisingly) the laplacian $\Delta_{\mathfrak{g}}$ on the euclidean space \mathfrak{g} with an extra constant term. The heat kernel of the flat laplacian is simply the gaussian kernel. Based on these relations, we can deduce, without much effort, the asymptotic expansion of the heat kernel of the laplacian on G .

(1.3) Notations Throughout this article G denotes a compact connected Lie group, and \mathfrak{g} its Lie algebra, namely, the tangent space $T_e G$ at the identity $e \in G$. We denote by \tilde{X} the left-invariant vector field on G generated by $X \in \mathfrak{g}$. We denote by $\langle \cdot, \cdot \rangle$ the (selected) bi-invariant metric on G . Such a metric is equivalent to an $\text{Ad}(G)$ -invariant inner product on \mathfrak{g} .

2. PRELIMINARIES

(2.1) We review some basic analytic and algebraic notions related to the laplacian on a compact Lie group.

Analytic Aspects.

(2.2) Proofs for most of the statements made in this subsection can be found in [2, Ch.2].

(2.3) **The Laplacian** Let $C^\infty(G)$ denote the space of smooth functions on G , and let $\mathfrak{X}(G)$ denote the space of smooth vector fields on G . Define the *gradient* operator $\text{grad} : C^\infty(G) \rightarrow \mathfrak{X}(G)$ and the *divergence* operator $\text{div} : \mathfrak{X}(G) \rightarrow C^\infty(G)$ by

$$\begin{aligned}\langle \text{grad } f, X \rangle_\kappa &= Xf, \\ (\text{div } X) \text{vol} &= L_X \text{vol},\end{aligned}$$

for $X \in \mathfrak{X}(G)$, where vol is the riemannian volume form and L_X is the Lie derivative with respect to X . Then the *laplacian* (or the *Laplace-Beltrami operator*) $\Delta_G : C^\infty(G) \rightarrow C^\infty(G)$ is defined by

$$\Delta_G f = \text{div}(\text{grad } f).$$

The definition of Δ_G is independent of local coordinates and depends only on the metric. Because our metric $\langle \cdot, \cdot \rangle$ is bi-invariant, so is the laplacian.

An expression for Δ_G in local coordinates (x_1, \dots, x_n) can be given as follows. Let g be the matrix defined by $g_{ij} = \langle \partial_i, \partial_j \rangle$ where $\partial_i = \partial/\partial x_i$. Let g^{ij} denote the (i, j) -entry of g^{-1} . Then,

$$(2.4) \quad \Delta_G f = \frac{1}{\sqrt{\det g}} \sum_{i,j} \partial_i (\sqrt{\det g} g^{ij} \partial_j f).$$

(2.5) **The Spectrum of the Laplacian** So far the laplacian is an unbounded operator whose domain $C^\infty(G)$ is a dense subspace of $L^2(G)$. The domain can be extended to the Sobolev space $H^2(G)$, that is, the space of measurable functions u on G such that the norm $\|u\|_{H^2} = \|u\|_{L^2} + \sum_i \|\partial_i u\|_{L^2} + \sum_{i,j} \|\partial_i \partial_j u\|_{L^2}$ is finite. The Sobolev embedding theorem tells us that $H^2(G)$ is a subspace of $L^2(G)$ and that the inclusion map is compact. The extension

$$\Delta_G : H^2(G) \rightarrow L^2(G)$$

is the unique self-adjoint extension of Δ_G which was originally defined on $C^\infty(G)$. In the language of the theory of unbounded operators, the laplacian is *essentially self-adjoint* on $C^\infty(G)$.

It turns out that $(1 - \Delta_G)$, where 1 denotes the identity operator, admits an inverse that is compact. Then the spectral theorem implies that the eigenfunctions $\{u_k\}_{k=1}^\infty$ of $(1 - \Delta_G)^{-1}$ form an orthonormal basis for $L^2(G)$. Owing to the regularity of elliptic differential operators, the eigenfunctions are of C^∞ . We can also conclude from the spectral theorem that the eigenfunctions u_k can be ordered in such a way that the corresponding eigenvalues $-\lambda_k$ of Δ_G give a nonincreasing unbounded sequence of negative real numbers,

$$0 > -\lambda_1 \geq -\lambda_2 \geq -\lambda_3 \geq \dots$$

The *heat diffusion operator* of Δ_G is then defined by the matrix

$$e^{t\Delta_G} = \begin{pmatrix} e^{-t\lambda_1} & & & \\ & e^{-t\lambda_2} & & \\ & & e^{-t\lambda_3} & \\ & & & \ddots \end{pmatrix}$$

with respect to the basis consisting of eigenfunctions of the laplacian.

(2.6) **The Heat Kernel** The heat diffusion operator is, in fact, an integral operator on $L^2(G)$ with a C^∞ -kernel. That means, there is some $K_t \in C^\infty(M \times M)$, such that

$$(2.7) \quad (e^{t\Delta_G} f)(x) = \int_G K_t(x, y) f(y) \text{vol}(y)$$

for any $f \in L^2(G)$. The kernel K_t is called the *heat kernel* of Δ_G . Owing to the equivariance of Δ_G , we have $K_t(x, y) = K_t(e, x^{-1}y)$, where e is the identity of G any x, y are arbitrary points in G . So the heat kernel is completely determined by the function

$$k_t(x) := K_t(e, x).$$

We will call this the *heat convolution kernel* for Δ_G . With it, equation (2.7) can be rephrased as

$$(2.8) \quad (e^{t\Delta_G} f)(x) = \int_G k_t(x^{-1}y) f(y) \text{vol}(y).$$

The trace of the heat diffusion operator, $Z(t) = \text{tr}(e^{t\Delta_G})$, is called the *partition function* or the *heat-trace* of Δ_G . It can be calculated in terms of the heat kernel as follows:

$$(2.9) \quad Z(t) = \int_G K_t(x, x) \text{vol}(x) = k_t(e) \text{vol}(G).$$

The convolution kernel k_t admits an *asymptotic expansion*

$$(2.10) \quad k_t \sim h_t(a_0 + a_1 t + a_2 t^2 + \dots)$$

as $t \rightarrow 0+$, where h_t is the gaussian kernel and a_i are smooth functions on M . The gaussian kernel h_t , under the exponential chart near $e \in G$, takes the form

$$h_t(X) = \frac{e^{-\|X\|^2/4t}}{(4\pi t)^{\dim M/2}}.$$

The asymptotic expansion (2.10) means that, for each nonnegative integers r, N , and n , there is a constant C such that

$$\left\| k_t - h_t \sum_{i=0}^N a_i t^i \right\|_{C^r} \leq C|t|^n$$

for sufficiently small t . Here $\|\cdot\|_{C^r}$ denotes the usual norm on $C^r(M)$. The asymptotic series $s_t := h_t \sum_{i=0}^{\infty} a_i t^i$ is a formal solution to the differential equation

$$(2.11) \quad (\partial_t + \Delta_G)s_t = 0$$

under the condition $s_t(e) = 1$. This gives a family of differential equations that can be solved inductively:

$$(\partial_t + \Delta_G)h_t \sum_{i=0}^k a_i t^i = h_t t^k \Delta_G a_k.$$

Algebraic Aspects.

(2.12) **The Universal Enveloping Algebra** The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of \mathfrak{g} is constructed by first taking the tensor algebra $T(\mathfrak{g})$ of \mathfrak{g} and then taking the quotient by the ideal $I(\mathfrak{g})$ generated by the elements of the form $X \otimes Y - Y \otimes X - [X, Y]$:

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I(\mathfrak{g}).$$

The adjoint action of $X \in \mathfrak{g}$ on \mathfrak{g} extends to $\mathcal{U}(\mathfrak{g})$ as a derivation. The invariant subspace of $\mathcal{U}(\mathfrak{g})$ under all such inner derivations is the center of the universal enveloping algebra;

$$\mathcal{Z}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}.$$

Suppose we have a simple tensor $X_1 \cdots X_n$ in $\mathcal{U}(\mathfrak{g})$. It generates the left-invariant differential operator $\tilde{X}_1 \cdots \tilde{X}_n$, where \tilde{X}_i denotes the left-invariant vector field on G generated by $X_i \in \mathfrak{g}$. This gives an algebra isomorphism between $\mathcal{U}(\mathfrak{g})$ and the space $D(G)$ of left-invariant differential operators. Under this bijection, the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra corresponds to the subalgebra of bi-invariant differential operators on G .

(2.13) **The Casimir Element** We pointed out earlier that the laplacian Δ_G is a bi-invariant operator. If $\{X_i\}_{i=1}^n$ ($n = \dim \mathfrak{g}$) is an orthonormal basis for \mathfrak{g} , then we claim that

$$(2.14) \quad \Delta_G = \sum_{i=1}^n \tilde{X}_i \tilde{X}_i.$$

In other words, the element in $\mathcal{U}(\mathfrak{g})$ that corresponds to Δ_G is the *Casimir element*:

$$\text{Cas} = \sum_{i=1}^n X_i X_i.$$

Since the differential operators on both sides of (2.14) are left-invariant, it is enough to check their equality at $e \in G$. To that end, take the exponential coordinate system (x_1, \dots, x_n) centered at e .

In other words, the coordinates (x_1, \dots, x_n) correspond to the point $\exp(\sum_{i=1}^n x_i X_i)$ in G . In this coordinate system, we have

$$(2.15) \quad \sum_{i=1}^n \tilde{X}_i \tilde{X}_i f \Big|_e = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} \Big|_0.$$

We need to show that the right-hand side is equal to $\Delta_G f$ at the identity; in other words, we need to verify that the expression of Δ_G at the identity under the exponential chart is $\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \Big|_0$. To see that this is the case, recall the riemannian exponential map $\text{Exp} : \mathfrak{g} \rightarrow G$ arising from the metric (see [6, Ch.1, §6] for details). This is, in general, different from the Lie-theoretic exponential map \exp which has nothing to do with the metric. But the two exponential maps do agree if the metric is bi-invariant, which can be seen as follows. Let ∇ be the riemannian connection so that it satisfies $\nabla_{\tilde{X}} \langle \tilde{Y}, \tilde{Z} \rangle = \langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle + \langle \tilde{Y}, \nabla_{\tilde{X}} \tilde{Z} \rangle$. Using the identity $2 \langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle = \tilde{X} \langle \tilde{Y}, \tilde{Z} \rangle + \tilde{Y} \langle \tilde{Z}, \tilde{X} \rangle - \tilde{Z} \langle \tilde{X}, \tilde{Y} \rangle + \langle [\tilde{X}, \tilde{Y}], \tilde{Z} \rangle - \langle [\tilde{Y}, \tilde{Z}], \tilde{X} \rangle + \langle [\tilde{Z}, \tilde{X}], \tilde{Y} \rangle$ and the skew-symmetry of the $\text{ad}(\mathfrak{g})$ -action, one can check that

$$(2.16) \quad \nabla_{\tilde{X}} \tilde{Y} = \frac{1}{2} [\tilde{X}, \tilde{Y}]$$

holds for all X, Y in \mathfrak{g} . In particular, $\nabla_{\tilde{X}} \tilde{X} = 0$. It follows [6, Ch.2, Prop.1.4] that the geodesic $\gamma_X(t)$, such that $\gamma_X(0) = e$ and $\gamma'_X(0) = X$, is a group homomorphism $\mathbb{R} \rightarrow G$. By the uniqueness of 1-parameter subgroups, we have $\gamma_X(t) = \exp(tX)$. This implies that the riemannian exponential map is identical to the Lie-theoretic exponential map. As a consequence, the matrix $[g_{ij}]$ of the metric under the exponential chart satisfies $g_{ij}(e) = \delta_{ij}$ (Kronecker delta) and $\partial_k g_{ij}(e) = 0$. Therefore, by equation (2.4), we have

$$(2.17) \quad \Delta_G f \Big|_e = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} \Big|_0.$$

Hence Δ_G agrees with $\sum_{i=1}^n \tilde{X}_i \tilde{X}_i$ at the identity, and this proves that these two invariant operators are equal everywhere on G .

(2.18) **The Duflo Isomorphism** Let $S(\mathfrak{g})$ be the symmetric algebra of \mathfrak{g} . The adjoint action of $X \in \mathfrak{g}$ on \mathfrak{g} extends, as an inner derivation, to $S(\mathfrak{g})$. Denote by $S(\mathfrak{g})^{\mathfrak{g}}$ the subalgebra of $S(\mathfrak{g})$ that is invariant under all such inner derivations. We identify $S(\mathfrak{g})^{\mathfrak{g}}$ with the constant coefficient differential operators on \mathfrak{g} . Duflo [3] showed that there is an algebra isomorphism

$$\text{Duf} : S(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathcal{Z}(\mathfrak{g}).$$

If we view, for the moment, $\mathcal{U}(\mathfrak{g})$ as the convolution algebra of distributions on G supported at $e \in G$ and $S(\mathfrak{g})$ as the convolution algebra of distributions on \mathfrak{g} supported at $0 \in \mathfrak{g}$, then Duf is $j \cdot \exp_*$, that is, the push-forward along the exponential map followed by the multiplication by the function $j(X) = \det^{1/2}(\frac{\sinh \text{ad}_X / 2}{\text{ad}_X / 2})$. We note that the push-forward map \exp_* alone gives the Poincaré-Birkhoff-Witt isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathcal{Z}(\mathfrak{g})$, which is only a vector space isomorphism.

3. THE ASYMPTOTIC EXPANSION OF THE HEAT KERNEL OF THE LAPLACIAN ON A COMPACT LIE GROUP

(3.1) Let $\Delta_{\mathfrak{g}}$ be the laplacian of the euclidean space \mathfrak{g} ; if $\{X_i\}_{i=1}^n$ is an orthonormal basis for \mathfrak{g} , then $\Delta_{\mathfrak{g}} = \sum_{i=1}^n X_i X_i$. This is an element of $S(\mathfrak{g})^{\mathfrak{g}}$, which we identify as the space of constant coefficient differential operators on \mathfrak{g} . The image of $\Delta_{\mathfrak{g}}$ under the Duflo isomorphism is

$$(3.2) \quad \text{Duf}(\Delta_{\mathfrak{g}}) = \text{Cas} + \frac{1}{24} \text{tr}_{\mathfrak{g}}(\text{Cas}),$$

where $\text{tr}_{\mathfrak{g}}$ denotes the trace for the linear operators on \mathfrak{g} obtained by extending the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Equation (3.2) can be proved in more than one way. For a Lie-algebraic proof, we refer to the work of Alekseev and Meinrenken

[1]. Recall that Cas , under the identification of $\mathcal{Z}(\mathfrak{g})$ with the space of bi-invariant differential operators on G , corresponds to the laplacian Δ_G on G . And, by a result of Kostant [10, Eq.1.85], we have

$$(3.3) \quad \frac{1}{24} \text{tr}_{\mathfrak{g}} \text{Cas} = -\langle \rho, \rho \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product—induced from the metric—on the dual space \mathfrak{t}^* of a maximal abelian subalgebra \mathfrak{t} of \mathfrak{g} , and $\rho \in \mathfrak{t}^*$ is the half the sum of the positive roots of G . Hence, we have

$$(3.4) \quad \text{Duf}(\Delta_{\mathfrak{g}}) = \Delta_G - \langle \rho, \rho \rangle.$$

(3.5) **Lemma** *Let $\text{Duf}(\Delta_{\mathfrak{g}})^{\text{exp}}$ be the differential operator defined near a neighborhood of $0 \in \mathfrak{g}$ by expressing the differential operator $\text{Duf}(\Delta_{\mathfrak{g}})$ on G under the exponential chart near the identity $e \in G$. We have*

$$\text{Duf}(\Delta_{\mathfrak{g}})^{\text{exp}} = j^{-1} \circ \Delta_{\mathfrak{g}} \circ j,$$

where j and j^{-1} above indicates the multiplication by the function $j(X) = \det^{1/2}(\frac{\sinh \text{ad}_X}{\text{ad}_X}/2)$ and its reciprocal, respectively.

Proof. Recall that the Duflo isomorphism is given by $j \cdot \exp_*$ under the identification of $\mathcal{U}(\mathfrak{g})$ with the convolution algebra of distributions on G supported at $e \in G$ and $S(\mathfrak{g})$ with the convolution algebra of distributions on \mathfrak{g} supported at $0 \in \mathfrak{g}$. Thus, back in the language of differential operators, $\text{Duf}(\Delta_{\mathfrak{g}})f|_e = \Delta_{\mathfrak{g}}(jf^{\text{exp}})|_0$ where $f^{\text{exp}} := f \circ \exp$. Since $j(0) = 1$, we have $\text{Duf}(\Delta_{\mathfrak{g}})^{\text{exp}}f^{\text{exp}}|_0 = (j^{-1} \circ \Delta_{\mathfrak{g}} \circ j)f^{\text{exp}}|_0$. Thus, $\text{Duf}(\Delta_{\mathfrak{g}})^{\text{exp}}$ agrees with $j^{-1} \circ \Delta_{\mathfrak{g}} \circ j$ at $0 \in \mathfrak{g}$. This implies that they agree on a neighborhood of $0 \in \mathfrak{g}$, provided that the operators $\text{Duf}(\Delta_{\mathfrak{g}})^{\text{exp}}$ and $j^{-1} \circ \Delta_{\mathfrak{g}} \circ j$ both define invariant differential operators near $e \in G$. The invariance of $\text{Duf}(\Delta_{\mathfrak{g}})^{\text{exp}}$ is clear from equation (3.4). For the invariance of $j^{-1} \circ \Delta_{\mathfrak{g}} \circ j$, we refer to [7, Ch.II, Eq.71, p.273]. \square

(3.6) **Lemma** *Let p_t be the convolution kernel of $e^{t \text{Duf}(\Delta_{\mathfrak{g}})}$. Then $p_t^{\text{exp}} := p_t \circ \exp$ has the asymptotic expansion*

$$p_t^{\text{exp}} \sim h_t j^{-1}, \quad t \rightarrow 0+,$$

valid in some neighborhood of $0 \in \mathfrak{g}$, where h_t is the gaussian kernel on \mathfrak{g} .

Proof. Let $s_t := h_t \sum_{i=0}^{\infty} a_i t^i$ be the asymptotic expansion for p_t^{exp} . It is the formal solution to

$$(3.7) \quad (\partial_t + \text{Duf}(\Delta_{\mathfrak{g}})^{\text{exp}})s_t = 0,$$

where $\text{Duf}(\Delta_{\mathfrak{g}})^{\text{exp}}$ is the differential operator $\text{Duf}(\Delta_{\mathfrak{g}})$ expressed in the exponential chart near the identity $e \in G$. By Lemma (3.5), the differential equation (3.7) is equivalent to

$$(\partial_t - j^{-1} \circ \Delta_{\mathfrak{g}} \circ j)s_t = 0.$$

We need to show that h_t/j satisfies this differential equation. This is easily done by invoking the fact that h_t satisfies the heat equation $(\partial_t - \Delta_{\mathfrak{g}})h_t = 0$; indeed,

$$(\partial_t - j^{-1} \circ \Delta_{\mathfrak{g}} \circ j)h_t/j = j^{-1} \partial_t h_t - j^{-1} \Delta_{\mathfrak{g}} h_t = j^{-1} (\partial_t - \Delta_{\mathfrak{g}})h_t = 0. \quad \square$$

(3.8) **Lemma** *Let G be a compact connected Lie group equipped with a bi-invariant metric. The scalar curvature S of G is equal to $-\frac{1}{4} \text{tr}_{\mathfrak{g}}(\text{Cas})$.*

Proof. We pointed out in (2.16) that the riemannian connection ∇ on G satisfies $\nabla_{\tilde{X}} \tilde{Y} = \frac{1}{2} [\tilde{X}, \tilde{Y}]$. A routine calculation shows that the Riemann curvature tensor Rm satisfies

$$\text{Rm}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = -\frac{1}{4} \langle [[X, Y], Z], W \rangle = -\frac{1}{4} \langle [X, Y], [Z, W] \rangle,$$

where, for the last equality, we have used the fact that $\text{ad}(\mathfrak{g})$ -action is skew-symmetric. Let $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$ be an orthonormal basis for \mathfrak{g} . For the scalar curvature S , we have

$$\begin{aligned} S &= \sum_{i,j=1}^{\dim \mathfrak{g}} \text{Rm}(\tilde{X}_i, \tilde{X}_j, \tilde{X}_j, \tilde{X}_i) = -\frac{1}{4} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle [[X_i, X_j], X_j], X_i \rangle \\ &= -\frac{1}{4} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle \text{ad}(\text{Cas})X_i, X_i \rangle = -\frac{1}{4} \text{tr}_{\mathfrak{g}}(\text{Cas}). \end{aligned} \quad \square$$

(3.9) **Theorem** *Let G be a compact connected Lie group equipped with a bi-invariant metric. Let k_t be the heat convolution kernel for the laplacian on G . Then $k_t^{\text{exp}} = k_t \circ \exp$ has the asymptotic expansion*

$$k_t^{\text{exp}} \sim \frac{h_t}{j} e^{tS/6}$$

as $t \rightarrow 0+$, valid in a neighborhood of $0 \in \mathfrak{g}$, where S is the scalar curvature, j is the function defined by the power series $j(X) = \det^{1/2}(\frac{\sinh \text{ad}_X/2}{\text{ad}_X})$ and h_t is the gaussian kernel on \mathfrak{g} , that is, $h_t(X) = e^{-\|X\|^2/4t}/(4\pi t)^{\dim \mathfrak{g}/2}$.

Proof. Owing to equations (3.3), (3.4), and Lemma (3.8), we have $e^{t\Delta_G} = e^{tS/6} e^{t\text{Duf}(\Delta_{\mathfrak{g}})}$. This implies $k_t = e^{tS/6} p_t$ where p_t is the convolution kernel for $e^{t\text{Duf}(\Delta_{\mathfrak{g}})}$. The theorem now follows from Lemma (3.6). \square

(3.10) **Corollary** *Let G be a compact connected Lie group equipped with a bi-invariant metric. Let S be the scalar curvature. The heat-trace $Z(t) = \text{tr}(e^{t\Delta_G})$ of the laplacian Δ_G on G has the asymptotic expansion*

$$Z(t) \sim \text{vol}(G) e^{tS/6}$$

as $t \rightarrow 0+$.

Proof. This follows from Theorem (3.9) and equation (2.9). \square

REFERENCES

- [1] A. Alekseev and E. Meinrenken, *Lie theory and the Chern-Weil homomorphism*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 2, 303–338, DOI 10.1016/j.ansens.2004.11.004 (English, with English and French summaries). MR2144989 (2006d:53020)
- [2] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original. MR2273508 (2007m:58033)
- [3] M. Duflo, *Opérateurs différentiels bi-invariants sur un groupe de Lie*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 2, 265–288 (French, with English summary). MR0444841 (56 #3188)
- [4] J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), no. 1, 39–79. MR0405514 (53 #9307)
- [5] L. Gårding, *Dirichlet's problem for linear elliptic partial differential equations*, Math. Scand. **1** (1953), 55–72. MR0064979 (16,366a)
- [6] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978. MR514561 (80k:53081)
- [7] ———, *Groups and geometric analysis*, Pure and Applied Mathematics, vol. 113, Academic Press Inc., Orlando, FL, 1984. Integral geometry, invariant differential operators, and spherical functions. MR754767 (86c:22017)
- [8] H. P. McKean Jr. and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry **1** (1967), no. 1, 43–69. MR0217739 (36 #828)
- [9] S. Minakshisundaram and Å. Pleijel, *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds*, Canadian J. Math. **1** (1949), 242–256. MR0031145 (11,108b)
- [10] B. Kostant, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. J. **100** (1999), no. 3, 447–501, DOI 10.1215/S0012-7094-99-10016-0. MR1719734 (2001k:22032)
- [11] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), no. 4, 441–479, DOI 10.1007/BF01456804 (German). MR1511670
- [12] ———, *Über die Abhängigkeit der Eigenschwingungen einer Membran und deren Begrenzung*, J. Reine Angew. Math. **141** (1912), 1–11, DOI 10.1515/CRL.1912.141.1 (German).

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802 U.S.A.
E-mail address: hong@math.psu.edu
URL: <http://www.math.psu.edu/hong>